Government of India
Ministry of Earth Sciences
India Meteorological Department
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# Lecture notes On Dynamic Meteorology For E-learning phase of Forecaster's Training Course

Prepared
By
Dr.Somenath Dutta

# Chapter-I

# Circulation and Vorticity

*Content: Definition & Mathematical expression of Circulation. Absolute and relative circulation. Circulation theorems. Interpretation of terms in the circulation theorem. Application of circulation theorems.*

***Circulation:***

Definition:

Circulation is defined as a macro-scale measure of rotation of fluid. Mathematically it is defined as a line integral of the velocity vector around a closed path, about which the circulation is measured.

Circulation may be defined for an arbitrary vector field, say, $\vec{B}$. Circulation '$C_B$' of an arbitrary vector field $\vec{B}$ around a closed path, is mathematically expressed as a line integral of $\vec{B}$ around that closed path, i.e., $C_B = \oint \vec{B}.d\vec{l}$ .

In Meteorology, by the term, 'Circulation' we understand the circulation of velocity vector. Hence, in Meteorology circulation around a closed path is given by C = $\oint \vec{V} . \vec{dl}$ ....(C1.1). From this expression it is clear that circulation is a scalar quantity.

Conventionally, sign of circulation is taken as positive (or negative) for an anticlockwise rotation (or for a clockwise rotation) in the Northern hemisphere. Sign convention is just opposite in the Southern hemisphere. Since we talk about absolute and relative motion, hence we can talk about absolute circulation and relative circulation. They are respectively denoted by $C_a$ and $C_r$ respectively and are defined as follows:

$C_a = \oint \overrightarrow{V_a} . \overrightarrow{dl}$ .... (C1.2)

and $C_r = \oint \overrightarrow{V_r} . \overrightarrow{dl}$ .... (C1.3)

Where $\overrightarrow{V_a}$ and $\overrightarrow{V_r}$ are the absolute and relative velocities respectively.

**Stokes Theorem:-**

It states that the line integral of any vector $\vec{B}$ around a closed path is equal to the surface integral of $\vec{\nabla}\times\vec{B}.\vec{n}$ over the surface 'S' enclosed by the closed path, where $\hat{n}$ is the outward drawn unit normal vector to the surface 'S'. So, $\oint \vec{B}\cdot\vec{dl} = \iint(\vec{\nabla}\times\vec{B})\cdot\hat{n}\,ds$.

***The Circulation Theorems:***

Circulation theorems deal with the change in circulation and its cause(s).

For an arbitrary vector field, $\vec{B}$ the circulation theorem states that the time rate of change of circulation of $\vec{B}$ is equal to the circulation of the time rate of change of $\vec{B}$, i.e.,

$$\frac{d}{dt}\oint \vec{B}.d\,l = \oint \frac{d\vec{B}}{dt}.d\vec{l} \quad \text{...........(C1.4)}$$

This theorem may be applied to the absolute velocity vector ($\vec{V}_a$) as well as to the relative velocity vector ($\vec{V}_r$).

***Kelvin's Circulation theorem:***

It is the circulation theorem, when applied to the absolute velocity ($\vec{V}_a$) of fluid motion.

So according to Kelvin's Circulation theorem,

$$\frac{d_a C_a}{dt} = \oint \frac{d_a \overrightarrow{V_a}}{dt}\cdot\overrightarrow{dl} \quad \text{.....(C1.5).}$$

Proof: We know that $C_a = \oint \overrightarrow{V_a}\,.\overrightarrow{dl}$

So, $\frac{d_a C_a}{dt} = \frac{d_a}{dt}\oint \vec{V}_a\,.d\vec{l}$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a\vec{V}_a}{dt}.d\vec{l} + \oint \vec{V}_a.\frac{d_a}{dt}(d\vec{l})$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a\vec{V}_a}{dt}.d\vec{l} + \oint \vec{V}_a.d_a\vec{V}_a$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a\vec{V}_a}{dt}.d\vec{l} + \oint d_a(\frac{\vec{V}_a.\vec{V}_a}{2})$

Or, $\frac{d_a C_a}{dt} = \oint \frac{d_a \vec{V}_a}{dt}.d\vec{l}$ , as the line integral of an exact differential around a closed path vanishes.

Conventionally, $\frac{d_a C_a}{dt}$ or $\frac{dC_r}{dt}$ are known as acceleration of circulation (absolute or relative).

So, in Meteorology, circulation theorem simply states that the acceleration of circulation is equal to the circulation of acceleration.

*A corollary to Kelvin's circulation theorem*:

We know that equation for absolute motion is given by,

$\frac{d_a \vec{V}_a}{dt} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}^* + \vec{F}$ ..........(C1.6), where symbols carry their usual significances.

Here, $\vec{g}^* = -\frac{GM}{r^2}\left(\frac{\vec{r}}{r}\right)$ is the gravitational attraction exerted by earth on a unit mass with position vector, $\vec{r}$ , with respect to the centre of the earth. It is clear that $\vec{g}^*$ is a single valued function of '$r$'. Also it is known that all force fields which are single valued functions of distance ($r$), are conservative field of forces. ('Dynamics of a particle', by S.L.Lony). Hence, $\vec{g}^*$ is a conservative force field. It is also known that work done by a conservative force field around a closed path is zero.

Hence, $\oint \vec{g}^*.d\vec{l} = 0$ ....(C1.7).

Again, from Stoke's law we know that for a vector field, $\vec{B}$ ,

$\oint \vec{B}.d\vec{l} = \iint_s \vec{\nabla} \times \vec{B}.\hat{n}\, ds$ ..........(C1.8)

Where S is the surface area enclosed by a closed curve, around which the circulation of $\vec{B}$ is measured, and '$\hat{n}$' is the outward drawn unit vector normal to the surface area S.

So, $\oint -\frac{1}{\rho}\vec{\nabla}p.d\vec{l} = \iint_s \vec{\nabla} \times \left(-\frac{1}{\rho}\vec{\nabla}p\right).\hat{n}\, ds = \iint_s \frac{\vec{\nabla}\rho \times \vec{\nabla}p}{\rho^2}.\hat{n}\, ds$ ..........(C1.9)

Hence, using (C1.6), (C1.7) and (C1.9) in (C1.5), we have for friction less flow,

$\frac{d_a C_a}{dt} = \iint_s \frac{\vec{\nabla}\rho \times \vec{\nabla}p}{\rho^2}.\hat{n}\, ds$ ..............(C1.10)

We know that in a barotropic atmosphere the density, $\rho$, is a function of pressure only, i.e., $\rho$ can be expressed as, $\rho = f(p)$.

Hence, $\vec{\nabla}\rho = f'(p)\vec{\nabla}p \Rightarrow \vec{\nabla}\rho \times \vec{\nabla}p = \vec{0}$, where, $\vec{0}$ is null vector.

Therefore, for a frictionless barotropic flow, $\frac{d_a C_a}{dt}$ =0…….(C1.11). This is a direct corollary to the Kelvin's theorem. Hence from Kelvin's circulation theorem it may be stated that for frictionless flow change in absolute circulation is solely due to the baroclinicity of the atmosphere.

Solenoidal vector and Jacobian:

Suppose, $A, B$ are two scalar functions. Then, Jacobian of these functions, is denoted by $J(A,B)$ and is given by,

$J(A,B) = \begin{vmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} \end{vmatrix} = \hat{k}.\vec{\nabla}A \times \vec{\nabla}B$. Also, $\vec{\nabla}A \times \vec{\nabla}B$ is called $A, B$ Solenoidal vector and is denoted by, $\vec{N}_{A,B}$.

So, the vertical component of solenoidal vector is the Jacobian.

Now, it will be shown that, $J(A,B)$ represents change in $A(x, y)$ along the isolines of $B(x, y)$ and vice-versa.

We have, $|J(A,B)| = |\vec{\nabla}A \times \vec{\nabla}B| = |\vec{\nabla}A||\vec{\nabla}B|\sin\theta$, where, θ is the angle between $\vec{\nabla}A$ and $\vec{\nabla}B$. We know that $\vec{\nabla}A$, $\vec{\nabla}B$ are normal to the isolines of $A, B$ respectively. Hence the angle between isolines of $A, B$ is also θ. If α is the angle between isolines of $B$ and $\vec{\nabla}A$, then θ = $90^0$ - α. So, $|J(A,B)| = |\vec{\nabla}A||\vec{\nabla}B|\cos\alpha$. Now, $|\vec{\nabla}A|\cos\alpha$ represents the magnitude of the projection of $\vec{\nabla}A$ on the isoline of $B$. As $\vec{\nabla}A$ represents change of $A$, hence it follows that $|\vec{\nabla}A|\cos\alpha$ represents the change of $A$ along the isolines of $B$. Thus, for a given gradient of $B$, $|J(A,B)|$ represents the change of $A$ along the isolines of $B$. Similarly, it can be shown that for a given gradient of $A$, $|J(A,B)|$ represents the change of $B$ along the isolines of $A$.

So, it is clear that as the magnitude of θ increases, the magnitude of the α = ($90^0$-θ) decreases and hence the magnitude of the Jacobian increases. It is zero when $\theta = 0^0$ and is maximum when $\theta = 90^0$.

**Barotropic and Baroclinic Atmosphere:**

Here we shall discuss the salient features of the solenoid vector.

Solenoid vector, denoted by $\overrightarrow{N}_{\rho,P}$ or $N_{T,p}$ is given by

$$\overrightarrow{N}_{\rho,P} = \vec{\nabla}\rho \times \vec{\nabla}p \text{ .... (C1.12) or}$$

$$\vec{N}_{T,p} = \vec{\nabla}T \times \vec{\nabla}p \text{ .... (C1.13).}$$

When the atmosphere is barotropic, then, there is no horizontal temperature gradient. Hence in such an atmosphere, $\vec{\nabla}T = \hat{0}$ [$\hat{0}$ is the null vector].

Hence in such an atmosphere, $\frac{R}{p}\left(\vec{\nabla}p \times \vec{\nabla}T\right) = \hat{0}$.

$\overrightarrow{N}_{T,p} = \hat{0}$ . Hence $\vec{\nabla}T\ ll\ \vec{\nabla}P$ .

Hence in such case, the isobars and isotherms (or the isolines of density $\rho$) are parallel to each other. This has been shown in fig.1.1.

But if the atmosphere is not barotropic, then these lines are no longer parallel, rather they intersect each other. Now, when they intersect, they form small rectangles like ABCD (shown in fig 1.2). Such rectangles are called solenoid. It is shown below that the magnitude of Solenoid Vector is equal to the number of solenoids formed in unit area in the vertical plane.

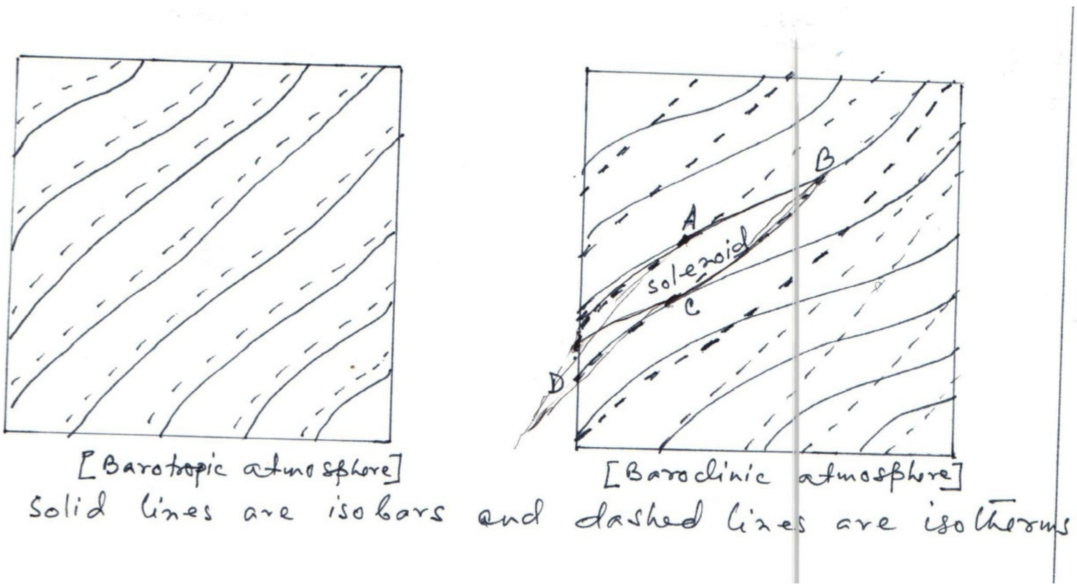



Fig.1.1 & 1.2

Area of a single solenoid ABCD $= a h_1$, where $a$ is the length of the side AB and $h_1$ is the length of the altitude DE, as shown in figure 1.2.

Now, $h_1 = b\sin\theta$, where, $b$ is the length of the side AD and $\theta$ is the angle between the sides AB and AD.

Hence, area of the solenoid ABCD $= ab\sin\theta$.

Now, $\left|\vec{\nabla}T \times \vec{\nabla}p\right| = \left|\vec{\nabla}T\right|\left|\vec{\nabla}p\right|\sin\theta = \frac{1}{h_2}\frac{1}{h_1}\sin\theta = \frac{1}{a\,b\sin\theta}$, where, $h_2$ is the length of the altitude BF.

So, area $ab\sin\theta$ is contained in 1 solenoid.

Hence, unit (= 1) area is contained in $\frac{1}{a\,b\sin\theta}$ numbers of solenoid. So, the magnitude of above solenoidal vector represents the number of solenoids in unit area in a vertical plane.

Practically the angle between isobar and isotherms gives a qualitatively measure of baroclinicity of the atmosphere. Because as the angles are smaller, the isobars and isotherms are very close to be parallel to each other i.e. the atmosphere is mostly barotropic. But as the angle increases, the isotherms and isobars become far away from being parallel to each other i.e. the atmosphere is mostly baraclinic. Also it is worth to note that as the angles between isotherms and isobars are smaller, numbers of solenoids are also smaller and if angle increases, the numbers of solenoids are also increases. So in the day to day charts to examine the qualitative measure of baroclinicity we need to estimate only the angle between isobars and isotherms or in the constant pressure chart we need to examine the angle between contour lines and the isotherms.

**Bjerknees Circulation Theorem:**

Kelvin's circulation theorem tells us about the change of absolute Circulation. But it is more important to know about the change of circulation with respect to the earth. Hence it is more important to know the change of relative circulation.

Bjerkness circulation theorem tells us about the change in relative circulation

According to Bjerkness circulation theorem, we have

$$\frac{dC_r}{dt} = \frac{dC_a}{dt} - 2\Omega\frac{dS_E}{dt} \qquad \text{………………….. (C1.14)}$$

Proof: We know that, $\vec{V}_a = \vec{V} + \vec{\Omega}\times\vec{r}$.

$$\Rightarrow \oint \vec{V}_a . d\vec{l} = \oint \vec{V}. d\vec{l} + \oint \left(\vec{\Omega}\times\vec{r}\right). d\vec{l}$$

$$\Rightarrow C_a = C_r + \iint_S \vec{\nabla}\times(\vec{\Omega}\times\vec{r}).\hat{n}\, ds$$ (Stoke's theorem used for 2nd line integral)

$$\Rightarrow C_a = C_r + \iint_S 2\vec{\Omega}.\hat{n}\, ds$$

Now, $\vec{\Omega}.\hat{n} = |\vec{\Omega}||\hat{n}|\cos(\vec{\Omega},\hat{n}) = \Omega \sin\phi$, where, $\phi$ is the latitude of the area element $ds$ and $\Omega = |\vec{\Omega}|$.

Hence, $\Rightarrow C_a = C_r + 2\Omega \iint_S ds Sin\phi = 2\Omega S_E$

Where, $S_E = \int dS_E = \int dS$ Sin$\phi$

and ds Sin$\phi$ is the area of the projection of ds on the equatorial plane, as shown in the figure below:

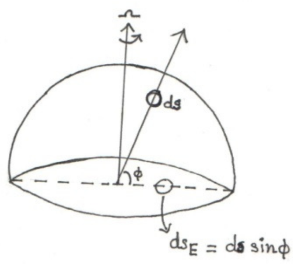


Fig - 2(a)

The first term $\frac{dC_a}{dt}$, have already been discussed in the Kelvin's circulation theorem. Now we shall discuss the 2nd term -2$\Omega \frac{dS_E}{dt}$.

Considering the effect of the 2nd term independently the Bjerkness circulation theorem gives us

$$C_{r2} - C_{r1} = -2\Omega(S_2 Sin\phi_2 - S_1 Sin\phi_1) \qquad \text{(C1.15)}$$

Where $C_{r1}$ = Initial relative circulation;

$C_{r2}$ = Final relative circulation

$S_1$ = Initial area enclosed by the closed path

$S_2$ = Final area enclosed by the closed path

$\phi_1$ = Initial Latitude

$\phi_2$ = Final Latitude

Thus the above equation tells us that the change in relative circulation may be due to

(i) change in area enclosed by the closed path

(ii) change in latitude

(iii) Non uniform vertical motion superimposed on the circulation

- Effect of the change in area enclosed by the closed path on the change in relative circulation :

If the area 'S' enclosed by the closed path increased from $S_1$ to $S_2$ , remaining at the same latitude '$\phi$', then the resulting change in relative circulation is given by

$C_{r2} - C_{r1} = -2\Omega Sin\phi \,(S_2 - S_1) < 0$, since, $S_2 > S_1$.

Thus Cyclonic circulation decreases as the area enclosed by the closed circulation increases. Physically it may be interpreted as follows:

Area enclosed by a closed circulation increases if and only if the divergence increases or convergence decreases. Then due to the Coriolis force the stream line turn anti-cyclonically or the already cyclonically turned streamlines turn less cyclonically. As a result of which cyclonic circulation reduces. Similarly due to convergence when the area enclosed by the circulation decreases, the cyclonic circulation increases. This has been shown in the figure below:

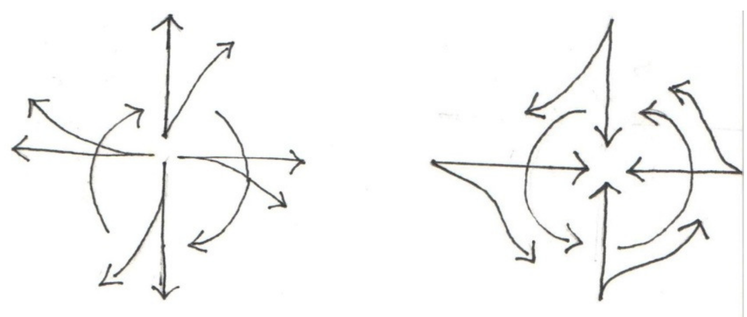


Fig- 2(b) Fig-2(c)

- Effect of the change in latitude on the change in relative circulation : Now suppose a circulation moves from a lower latitude $\phi_1$ to a higher latitude $\phi_2$, without any change in the area enclosed by the circulation. Then the resulting change in the relative circulation is given by

$$C_{r2} - C_{r1} = -2\Omega S(Sin\phi_2 - Sin\phi_1) < 0$$

Since $Sin\phi_2 > Sin\phi_1$

Hence a circulation loss its cyclonic circulation as it moves towards higher latitude.

Similarly it can be shown that when a cyclonic circulation moves towards lower latitude, then it gains cyclonic circulation.

- Effect of imposition of non uniform vertical motion on the change in relative circulation.

Now consider a different situation, when neither the area enclosed by the circulation changes nor the cyclonic circulation moves, but non uniform vertical motion is applied to the closed circulation. Then the inclination of the plane of rotation of circulation with the equatorial plane changes, (shown in figure 1.6) as a result of which $S_E$ changes which leads to a change in $C_r$. This effect is known as *TIPPING EFFECT*.

***A possible explanation of sea/land breeze and thermally direct circulation using Kelvin's circulation Theorem:***

Sea breeze takes place during day time when ocean is comparatively cooler than land. Hence temperature increases towards land and also we know that temperature decreases upward. (i.e. increased downward). Thus the temperature gradient $\vec{\nabla} T$ is directed downward to the land. For the shake of simplicity we assume that pressure over land and sea is same, but it increases downward. Hence pressure gradient $\vec{\nabla} p$ is directed downward. Hence $\vec{\nabla} p \times \vec{\nabla} T$ gives the circulation in the direction from $\vec{\nabla} p$ to $\vec{\nabla} T$. Also the change in circulation pattern is given by $\vec{\nabla} p \times \vec{\nabla} T$. Hence if initially there was no circulation, then the above mentioned pressure and temperature pattern will generate a circulation directed from $\vec{\nabla} p$ to $\vec{\nabla} T$, which gives low level flow from ocean to land and in the upper level from land to ocean. This is nothing but sea breeze. Similarly land breeze and any thermally driven circulation pattern may be explained qualitatively.

## Chapter-2

*Content: Pressure tendency equation (No derivation): physical interpretation, in detail, of each term, representing different mechanisms of pressure change. Different isobaric patterns and their movement.*

By the term pressure tendency we mean in-situ or local time rate of change of pressure at a place. Operationally we represent it by P24P24 on our change chart. This is a very important parameter, used to identify the location of centre of a low pressure system, because we know that centre of a low pressure system moves in a direction where fall of pressure is maximum. Mathematically pressure tendency is denoted by $\frac{\partial \mathrm{p}}{\partial \mathrm{t}}$. Pressure tendency equation tells us different dynamical processes/mechanisms leading to change in pressure at a point.

Before going to the discussion, intuitively, let us first understand how pressure at a point (not necessary at surface only) in the atmosphere can change. For that let us consider an atmospheric air column, of unit cross section, extending from that point to infinity. We know weight of the air in this column is the atmospheric pressure at that point. Now the pressure at this point may change if the weight of the air in the column is changed, which may in turn is possible if the mass of air in the column can be changed. Now mass of air inside the column may change if relatively denser/rarer air comes and mixes with air inside the column or/and if there is inflow/outflow of air to/from the column or/and if air enters into/exit from the column across the base of the column. First one is technically known as vertically integrated lateral advection of density, second one is known as vertically integrated divergence/convergence and the last one is known as vertical motion at the point. Accordingly, pressure tendency at a point is determined by the combined effect of above.

Pressure tendency equation is given by

$$\frac{\partial p}{\partial t} = -g\int_{z_0}^{\infty} \rho(\vec{\nabla}_h.\vec{V}_h)dz + g\int_{z_0}^{\infty}\left(-\vec{V}_h.\vec{\nabla}_h\rho\right)dz + g\rho(z_0)w(z_0)$$

Left hand side of the above equation represents pressure tendency at a point at level $z = z_0$ and right hand side consists of three terms each of which representing some mechanisms for pressure change.

First term is known as divergence term. It represents net lateral divergence or convergence across the sidewall of an atmospheric column with base at $z = z_0$ and extending up to top of the atmosphere. We know that pressure at $z = z_0$ is nothing but the weight of air contained in an atmospheric column with base at $z = z_0$ having unit cross sectional area and extending up to top of the atmosphere. Now this weight will increase or decrease if mass of air inside this column increases or decreases. Again mass of air inside this column increases or decreases if there is net inflow (convergence) or out flow (divergence) of air. Hence, net lateral divergence leads to fall in pressure and net lateral convergence leads to a rise in pressure. For synoptic scale system, this term contributes significantly towards pressure change.

Second term expresses the net lateral advection of mass into the atmospheric column with base at $z = z_0$ having unit cross sectional area and extending up to top of the atmosphere. Clearly net positive advection leads to an increase in mass, which in tern leads to rise in pressure and net negative advection leads to a decrease in mass which in tern leads to fall in pressure.

Third term expresses flux of mass into the above atmospheric column across its base at $z = z_0$.

**Movement of different pressure systems:** Here we shall discuss the movement of pressure systems (lows/highs) for different isobaric patterns. Mainly we shall discuss Sinusoidal pattern, circular pattern and circular pattern beneath a Sinusoidal pattern above.

***Sinusoidal isobaric pattern***: Let us refer to the adjoining sinusoidal pressure pattern. Ahead of the trough there is divergence and ahead of the ridge there is convergence at the surface. Hence fall in pressure takes place ahead of trough and rise in pressure ahead of ridge. Due to this, after some time lowest pressure will be found ahead of trough, as a result trough will be shifted towards east of its present location. Hence, the pressure system will move in a westerly direction.

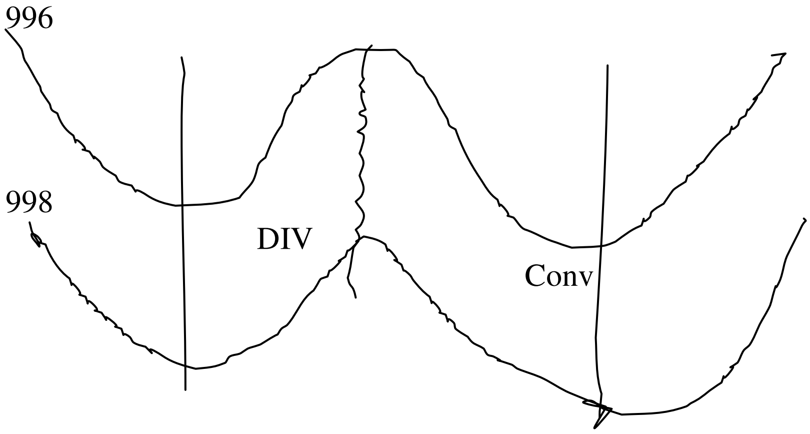



Fig.3

***Circular low-pressure pattern***: Let us consider the adjoining circular low-pressure pattern. Lowest pressure is at the center of the circular pattern. To the north of the center Coriolis force is higher than that to the south. As we know that Coriolis force makes flow anticyclonic, hence cyclonic wind will be more to the south than to the north of the center. Hence to the east of the center there is downstream decrease in wind speed and to the west there is down stream increase in wind speed. Hence divergence takes place to the west of the center as a result of which there will be fall in pressure to the west of the center. Due to this, center of low after some time will be shifted to the west of its present position. Hence net result is movement of the pressure system in an easterly direction.

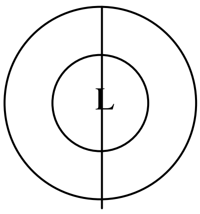



Fig-4

**Chapter-3.**

*Content: General definition of wave. Wave- amplitude, frequency, wave length, wave number. Definition & concept of Phase velocity, group velocity, Dispersion relation, dispersive wave, non- dispersive wave.*

Wave may be defined as a form of disturbance in a medium. When a disturbance is given to a part of an elastic medium, then that part gets displaced from its original position. But by the virtue of elasticity, a restoring force is developed in the displaced part, which helps it to return to its original position. This leads to an oscillatory motion, which is known as wave.

***Some useful concepts on waves:***

WAVE LENGTH:

It is defined as the distance between two consecutive points on the wave, which are in the same phase of oscillation, i.e. distance between two successive troughs or ridges.

WAVE NUMBER:

Wave number of a wave with wave length 'L' is defined as the number of such waves exist around a circle of unit radius. Hence wave number $k$ is defined by, $k = \frac{2\pi}{L}$, where L is the wave length.

Since a wave may travel in any direction, hence we may define wave length / wave number for three directions, viz. along x, y and z directions.

If $L_x, L_y$ and $L_z$ are respectively the wave lengths along x, y and z directions and if $k, l$ and $m$ are wave numbers along x, y and z directions, then

$$k = \frac{2\pi}{L_x}, l = \frac{2\pi}{L_y} \text{ and } m = \frac{2\pi}{L_z}.$$

FREQUENCY :

It is the number of wave produced in one second.It is denoted by ν.

PHASE VELOCITY:

We know that any disturbance behaves like a carrier. So, wave may be thought of as a carrier. Phase velocity is defined as the rate at which momentum is being carried by the wave. For practical purpose, it may be taken as the speed with which a trough / ridge moves.

It can be shown that, phase velocity in any direction = frequency / wave number in that direction. Thus if the phase velocity vector $\vec{C}$ has components $C_x$, $C_y$, $C_z$ along x, y and z direction, then $C_x = \nu / k, C_y = \nu / l$ and $C_z = \nu / m$.

GROUP VELOCITY :

It is the rate at which energy is being carried by the wave. When a single wave travels then the energy and momentum are carried by the wave at the same rate. But when a group of wave travel then momentum propagation rate and energy propagation rates are different. So, in such case group velocity and phase velocity are different. Thus if the phase velocity vector $\vec{C}_G$ has components $C_{GX}$, $C_{GY}$, $C_{GZ}$ along x, y and z direction, then $C_{GX} = \dfrac{\partial \nu}{\partial k}, C_{GY} = \dfrac{\partial \nu}{\partial l}$ and $C_{CZ} = \dfrac{\partial \nu}{\partial m}$.

DISPERSION RELATION :

It is a mathematical relation $\nu = f(k, l, m)$ between the frequency ( ν ) and wave numbers $k, l, m$.

Generally for any wave, phase velocity and group velocity is obtained from the dispersion relation.

If for any wave phase velocity and group velocity are same, then it is called a non-dispersive wave, otherwise it is a dispersive wave.

## Chapter-4.

*Content: General definition of hydrodynamic instability. Categorization of hydrodynamic instability in different ways.*

**Definition**: A mean flow field is said to hydro dynamically unstable if a small perturbation, introduced into the mean flow, grows spontaneously by extracting energy from the mean flow.

**Classification of hydrodynamic instability**:

Categorization of hydrodynamic instability may either be based on the state of the mean flow or on the mode of perturbation introduced.

Based on the former, hydro dynamic instability may be dynamic or static according as the mean flow is there or not. While discussing Barotropic instability, Baroclinic instability or Inertial instability, we always consider a mean flow having some speed. These are examples of dynamic instabilities. But while discussing Brunt Vaisala instability, we need not to take care of the mean flow. This is example of static instability.

Based on the later, hydro dynamic instability may be of two types, viz., parcel instability and wave instability. Sometimes perturbation may be introduced as a displacement to an air parcel and it is examined under what condition the parcel is moving away from its mean position. This is known as parcel instability. Brunt-Vaisala instability and Inertial instability are examples of parcel instability. In another case, the perturbation is given in the form of a wave super imposed on a mean flow and examined under what conditions the wave is being amplified. This is known as wave instability. Barotropic and Baroclinic instabilities are examples of wave instability.

The above categorization is shown below in a tabular form:

| **Hydrodynamic Instability** | | | |
|---|---|---|---|
| **Based On The State Of Mean Flow** | | **Based On The Mode Of Perturbation** | |
| Static Instability<br>Example: Brunt Vaisala instability. | Dynamic Instability<br>Examples: Inertial, Barotropic, Baroclinic | Parcel Instability<br>Inertial, Brunt Vaisala | Wave Instability<br>Barotropic, Baroclinic |

## Chapter-5

*Content: A brief introduction to PBL: Definition of PBL. Importance of PBL. Characteristics of PBL: the turbulent motion. Types of turbulent motion: Convective turbulence and Mechanical turbulence. Conditions, favourable for Convective turbulence and Mechanical turbulence. Depth of PBL and its diurnal and seasonal variation at a place. Different sub layers in PBL.*

**A Brief essay on PBL**:

PBL is the lower most portion of the atmosphere, adjacent to the earth's surface, where maximum interaction between the Earth surface and the atmosphere takes place and thereby maximum exchange of Physical properties like momentum, heat, moisture etc., are taking place.

Exchange of physical properties in the PBL is done by turbulent motion, which is a characteristic feature of PBL. Turbulent motion may be convectively generated or it may be mechanically generated.

If the lapse rate near the surface is super adiabatic, then PBL becomes positively Buoyant, which is favourable for convective motion. In such case PBL is characterized by convective turbulence. Generally over tropical oceanic region with high sea surface temperature this convective turbulence occurs. If the lapse rate near the surface is sub adiabatic then the PBL is negatively buoyant and it is not favourable for convective turbulence. But in such case, if there is vertical shear of horizontal wind, then Vortex (cyclonic or anti cyclonic) sets in, in the vertical planes in PBL, as shown in the adjacent fig 2b. This vortex motion causes turbulence in the PBL, known as mechanical turbulence.

If the PBL is positively buoyant as well as, if vertical shear of the horizontal wind exists, then both convective and Mechanical turbulence exits in the PBL. The depth of the PBL is determined by the maximum vertical extent to which the turbulent motion exists in PBL. On average it varies from few cms to few kms. In case of thunderstorms PBL may extend up to tropopause.

Generally at a place on a day depth of PBL is maximum at noon and in a season it is maximum during summer.

**Division of the PBL into different sub layers:**

The PBL may be sub divided into three different sections, viz viscous sub layer, the surface layer and the Ekmann layer or entrainment layer or the transition layer.

Viscous layer is defined as the layer near the ground, where the transfer of physical quantities by molecular motions becomes important. In this layer frictional force is comparable with PGF.

The surface layer extends from z = $z_0$ (roughness length) to $z = z_s$ with $z_s$, the top of the surface layer, usually varying from 10 m to 100 m. In this layer sub grid scale fluxes of momentum (eddy stress) and frictional forces are comparable with PGF.

The last layer is the Ekmann layer is defined to occur from $z_s$ to $z_i$, which ranges from 100 m or so to several kilometers or more. Above the surface layer, the mean wind changes direction with height and approaches to free stream velocity at the height z as the sub grid scale fluxes decrease in magnitude. In this layer both the COF and Eddy stress are comparable with PGF.

## Chapter-6

*Content: Energetics aspects of General circulation: Definition of Atmospheric energetics. Different form of atmospheric energies, viz., internal energy, potential energy and kinetic energy. Expressions for internal energy.*

By the term atmospheric energetics, we understand the different forms of energy, the atmosphere possesses and conversion between them.

Atmosphere possesses energy mainly in three forms, Viz., the internal energy, the kinetic energy and potential energy.

Atmospheric internal energy: It is due to heating of the atmosphere. To obtain an expression for global atmospheric internal energy, let us consider unit mass at temp $T^0$ K. Then internal energy of this unit mass is $C_V T$. Now consider an infinitesimal volume '$d\sigma$' with density '$\rho$' of the atmosphere. This volume is so small that the density $\rho$ practically remains invariant in it. So its mass is $\rho d\sigma$. So, the internal energy of this infinitesimal volume is $C_V T\rho d\sigma$.

Hence the internal energy of the global atmosphere is $\iiint_\sigma \rho C_V T d\sigma = I$ , say.

Atmospheric potential energy: It is due the vertical position of the centre of gravity of the atmosphere. The potential energy of unit mass at a height 'z' above the mean sea level is $gz$. Hence following the same argument as in I.E, we have the expression for potential energy of global atmosphere as $\iiint_\sigma g\rho z d\sigma = P$, say.

Atmospheric kinetic energy: The kinetic energy of the atmosphere is due to different atmospheric motion. Kinetic energy of an unit mass moving with velocity '$\vec{v}$' is $\frac{\vec{v}.\vec{v}}{2}$. Hence the expression for kinetic energy for global atmosphere is $\iiint_\sigma \rho \frac{\vec{v}.\vec{v}}{2} d\sigma$ =K, say.

Government of India
Ministry of Earth Sciences
India Meteorological Department
Meteorological Training Institute

# Lecture notes On Numerical Weather Prediction For E-learning phase of Forecaster's Training Course

Prepared
By
Dr.Somenath Dutta

## Chapter-1

*Content: History of NWP. Hierarchy of NWP models. What is broadly understood by the term 'NWP model'? Different components of a NWP model (basic concept only). Definition of Initial & boundary value problem (IVP & BVP). NWP as IVP+BVP.*

### 1. An Overview of Numerical Weather Prediction:

The main goal of a Meteorological office is to issue weather forecasts in different time scales to various user agencies like aviation, marine, agriculture, water management, builders, tourism industry, planners etc. and to general public. The forecast requirement varies from detailed weather forecasts in time scales of a few hours to days and to more general indication of the broad weather patterns of succeeding months, seasons or even beyond. For example, aviation industry needs weather information in time scales of a few hours or a day, whereas agriculture sector demands weather forecasts in time scales of a week. Weather forecasts of a month or a season in advance are required mostly by planners. Thus, the weather forecasts are broadly classified as (i) short-range (forecast validity upto 3 days), (ii) medium range (forecast validity beyond 3 days upto 10 days) and (iii) long-range (forecast validity beyond 10 days or a few weeks or a month or a season or even beyond).

Weather forecasting basically consists of two steps. The first step is to have an accurate assessment of the present/initial state of the atmosphere. This helps in identifying the different weather systems and their horizontal and vertical state. As we know, there are a variety of phenomena occurring in the atmosphere having different space and time scales. The characteristic sizes of these motions vary from a fraction to centimetre to several thousands of kilometres, with time scales of a fraction of a section to several weeks. Each of the various scales of motions has a varying degree of influence upon all the others and it is important to properly observe, analyse and account them in atmospheric studies and weather forecasting. As weather has no political boundary, one needs weather data from a fairly large region; the area from which data required increases with the duration of forecast made, since weather systems from one part of a region may travel and affect the weather condition over a far off region in course of time.

The second step in weather forecasting is to utilise a suitable technique to predict the future state of the atmosphere. Primarily, there are three methods (i) synoptic method (ii) statistical method and (iii) numerical weather prediction (NWP). In the synoptic method, a forecaster attempts to predict the future changes in the state of atmosphere from its initial state using his theoretical knowledge and experience. Here, various weather charts, commonly known as synoptic charts are analysed at a fixed time to understand the three dimensions of the atmospheric state. This method can generate forecast for a broad period over a broad region, can't generate time and location specific objective forecast. That way, this method is subjective and the success of the forecast depends heavily on the knowledge, experience and skill of the forecaster. The synoptic method is widely used in short range weather forecasting, especially in tropics.

The statistical technique is based on correlation and regression analysis. By analyzing the past weather records for a long period, useful relationship can be established relating the occurrence of the one weather event with another or a number of other weather elements. For example, 'All-India Rainfall' behaviour of southwest monsoon is related with El-Nino, Southern Oscillation, Winter Snow Cover over Himalayas, Surface Pressure over Northwest Australia etc. This technique is mostly used in long range forecasting.

The third method of weather forecasting which is more promising and has become more popular in recent decades is the NWP. In this method, predictions are made by solving the hydro-dynamical equations which govern the atmospheric motions using powerful computers. The method being objective, has gained tremendous boost with the progress in technology – observational, telecommunication and computer, and understanding of various atmospheric processes. While the synoptic method has inherent difficulties to provide realistic weather forecasts beyond 2-3 days, the NWP technique can be expected to provide objective forecasts for much longer time periods.

Currently, Forecast Services are based on conventional Synoptic Methods supplemented by use of Numerical Weather Prediction products.

**2. Historical Background NWP and hierarchy of NWP models**

From the historical point of view, V. Bjerknes was the first scientist who suggested in 1904 that the atmospheric prediction problem is an initial value marching problem. The first practical attempt to solve the set of equations that govern the atmospheric motions numerically was made by Lewis F. Richardson during the first world war. He used a desk calculator to compute the surface pressure tendencies at two grid points. Unfortunately, his results were in error by an order of magnitude and were totally unacceptable. As a result, his monumental work was ignored for more than two decades. The failure of Richardson's numerical treatment was, at that time, attributed to poor initial data available, especially the absence of upper air data. Later, it was discovered that the atmospheric equations in its complete form, so called 'primitive' form, admit solutions corresponding to not only the slow moving atmospheric waves (Rossby Waves) but also fast moving sound and gravity waves. These high speed waves amplify spuriously with the time and mask the solutions relating to atmospheric waves if not properly controlled. It was theoretically shown by Courant, Friedrichs and Lewy in 1928 that the space and time increments used in the marching scheme have to satisfy the stability criterion given by

$C (\Delta t/\Delta x) \leq 1$

where $C$ = speed of fastest wave

$\Delta t$ = time increment

$\Delta x$ = space increment

Later, in 1948, Charney showed that by making use of hydrostatic and geostrophic assumptions the high speed sound and gravity waves can be effectively 'filtered'. In 1950, using the first electronic calculator ENIAC and the filtered model, Charney, Fjortoft and Von Neumann produced the first successful numerical prediction.

Since then, there has been a rapid progress in all phases of NWP. These improvements are mainly due to considerable increase in the quantity of meteorological data advances in telecommunication system tremendous progress in computer technology and development of much better and sophisticated numerical models.

Even then the success of NWP is relatively poor in tropics compared to that in extra tropical region due to certain constraints peculiar to tropics. The major problems in the tropical belt are:

Sparse data network, as the region is dominated by the oceans, deserts and mountains from where observations on regular basis are poor and hence, difficulty in defining an accurate initial state.

Weak wind-pressure balance (small coriolis force and weak pressure gradient).

Dominance of meso-scale systems like cumulus convection, thunderstorms etc. which are known to supply energy to large scale systems like cyclones, depressions etc. but difficult either to detect by synoptic network or forecast well in advance.

Lack of a precise understanding and modelling of various physical processes like air-sea interaction, boundary layer forcing, radiation, convection etc.

Fortunately in the modern days with the advances in observational technology, telecommunication system, computing powers and understanding of different physical processes there is much needed progress in the field of NWP and the reliability of numerical forecasts is much higher compared to that of a few decades earlier.

Barotropic models do not allow temperature advection because wind is parallel to isotherms and hence, can’t forecast the development of new weather systems. They, in fact only extrapolate the system by advection of vorticity. For more accuracy and reliability, prediction of development of new systems is essential. For this purpose, thermal advection has to be included in the numerical model and hence more than one level is required to be incorporated in the model, i. e. one has to consider a baroclinic model. Accordingly 2-layer Baroclinic model ( by Charney and Eady) developed.

The two level baroclinic model has not proved to be a very successful prediction model primarily because it tends to produce stronger baroclinic development than observed in many cases. This was attributed to the fact that the heights of only two widely separated isobaric surfaces reveal very little of the detailed vertical structure of the atmospheric flow. However, the simplicity of this model does make it a useful tool for analysis of the physical processes occurring in baroclinic disturbances-extratropical frontal systems. The two level baroclinic model can be extended to multi-level model for greater vertical resolution and improved weather forecast.

Barotropic and 2 or multi layer baroclinic models are known as filtered model. However, filtered models, though filter out gravity waves and become easier to manage by numerical schemes, have a definite limit on their accuracy due to the various approximations made. Further, these models can't be used in low latitudes where quasi-geostrophic assumption is not valid. Under these circumstances, it was thought that direct utilisation of basic hydrodynamic equations in their primitive form might improve the accuracy of numerical models in general. The availability of much faster computers and improved understanding of computational problems made it possible to carryout time integration for longer period of the basic primitive equations using small time space. The first successful experiments using primitive equations were carried out by Hinkelman in 1959. Since then NWP has marched steadily forward and now a days primitive equations models are widely utilised world over for operational numerical weather prediction in different time scales.

**3. Different components of a NWP model:**

We know that weather $'\mathrm{WX}'$at a place $(\hat{\mathrm{x}})$ at a time $'\mathrm{t}'$ is a function of the basic meteorological parameters, viz., zonal wind (u), meridional wind (v), vertical wind (w), pressure (p), temperature (T), specific/relative humidity (q) etc at that time over that place. Mathematically, $\mathrm{WX} = \mathrm{WX}(\mathrm{u}(\hat{\mathrm{x}},\mathrm{t}), \mathrm{v}(\hat{\mathrm{x}},\mathrm{t}), \mathrm{w}(\hat{\mathrm{x}},\mathrm{t}), \mathrm{P}(\hat{\mathrm{x}},\mathrm{t}), \mathrm{T}(\hat{\mathrm{x}},\mathrm{t}), \mathrm{q}(\hat{\mathrm{x}},\mathrm{t}))$. So, if the values of these basic meteorological parameters at a place $(\hat{\mathrm{x}})$ at a time $'\mathrm{t}'$ are known, then by establishing a suitable functional relation, as above, weather at the place $(\hat{\mathrm{x}})$ at time $'\mathrm{t}'$ can be determined. Thus the problem of forecasting weather has been translated to the problem of forecasting above meteorological parameters. Now question is whether it is possible to predict these parameters at a place for future. The answer to this question lies on the fact, whether these variables satisfy following type equation or not:

$$\frac{\partial \mathrm{f}(\hat{\mathrm{x}},\mathrm{t})}{\partial \mathrm{t}} = \mathrm{g}(\hat{\mathrm{x}},\mathrm{t});\ \text{given } \mathrm{f}(\hat{\mathrm{x}},\mathrm{t}_0) = \mathrm{f}_0(\hat{\mathrm{x}})$$

This class of equations, in mathematical literatures, are known as initial value problems (IVP). Fortunately all the above mentioned variables satisfy such initial value problem as given below:

1. U-momentum equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial x} - 2\Omega(w\cos\varphi - v\sin\varphi) + \nu\nabla^2 u$$

2. V-momentum equation

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\varphi + \nu\nabla^2 v$$

3. W-momentum equation

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} - v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\varphi + \nu\nabla^2 w$$

4. Pressure tendency equation

$$\frac{\partial p(z)}{\partial t} = -\int_z^\infty g\rho(\vec{\nabla}.\vec{V}_h)dz - \int_z^\infty g(\vec{V}_h.\vec{\nabla}\rho)dz + g\rho(z)w(z)$$

5. Thermodynamic energy equation

$$\frac{\partial T}{\partial t} = \frac{1}{C_P}\frac{dQ}{dt} - u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} + \omega\sigma$$

6. Mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]$$

7. Moisture continuity equation

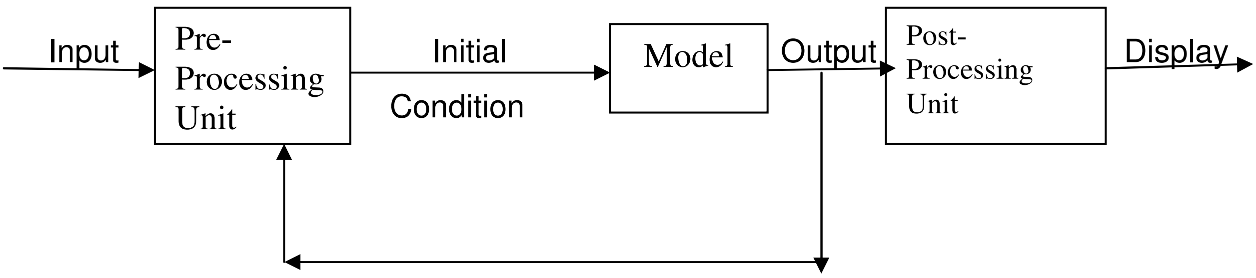
$$\frac{\partial q}{\partial t} = -u\frac{\partial q}{\partial x} - v\frac{\partial q}{\partial y} - w\frac{\partial q}{\partial z}$$

8. Equation of state

$$p = \rho RT$$

The above equations can be solved for future evolution of these variables, provided, values of these variables are known at initial time. In NWP, above equations are integrated forward with respect to time and the constants of integration are computed using the initial values of the variables.

Thus, without having the complete knowledge about these variables at initial time, we can't have a particular solution for future values of these variables. In a typical NWP system, broadly there are three compartments, viz., pre-processing unit, model and the post processing unit, as shown schematically below:




In the pre-processing unit, observed data, available at unevenly spaced observing points, are subjected to different quality control checks, spatial and temporal consistency check and climatology check etc, followed by a sophisticated interpolation scheme to prepare values of the above variables at different grid points. It is most likely that grid point data, prepared in this way may contain errors, which is removed/minimised within a given tolerance limit and thus the initial values of the variables at grid points, i.e., initial conditions are prepared.

With above initial conditions, the equations are numerically integrated forward with time, in the Model. After integration, future values of the above variables at different grid points are generated, known as output.

These raw output may be of very little use for the stake holders, rather, stake holder may like to have specific weather information, like, rainfall, visibility, divergence, vorticity, precipitable water content etc, which are prepared by post processing of raw output from model. This post processing is being done in post processing unit.

**4. Initial value problem (IVP) and Boundary value problem:** Problems of solving following types of equation

$\frac{\partial f(\hat{x},t)}{\partial t} = g(\hat{x},t)$; given $f(\hat{x},t_0) = f_0(\hat{x})$; where $g(\hat{x},t)$ is a known function, are known as initial value problems.

In mathematics, in the field of differential equations, a boundary value problem is a differential equation together with a set of additional constraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation which also satisfies the boundary conditions.

Example: Solve $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = g(x,y)$, where, $g(x,y)$ is a known function and given the following boundary condition:

$$\varphi(x,-b) = \varphi_{-b}, \varphi(x,b) = \varphi_b, \varphi(-a,y) = \varphi_{-a}, \varphi(a,y) = \varphi_a$$

As the problem of forecasting weather requires the complete information about the present state of the atmosphere, hence it is essentially an IVP. At the same time as the weather condition at a place at a given time is also dependent on the weather condition over the surrounding of the place also. So, this is a BVP also. Thus problem of forecasting weather at a place is essentially an IVP as well as a BVP.

## Chapter-2

*Content: Basic concepts of objective analysis & initialization.*

**Basic concept of objective analysis & Initialisation:** We know that in NWP the governing equations, which are essentially non linear partial differential equations, are integrated forward in time to obtain the future values of the field variables.

Since the governing equations are non linear partial differential equations and as there is no method to integrate analytically such equation, these equations are integrated numerically. To integrate the equations numerically the analytical time and spatial domain are replace by set of grid points. As the solving of these non linear PDE are BVP (boundary value problem) and IVP, (Initial value problem) we require the values of the field variable at all grid points at each level at initial time. But the observed field variables are not necessarily at the grid points. Thus our first task is to prepare the values of the field variable at grid points from the observing points, using some interpolation method. This is known as objective analysis. Different types of objective analysis scheme are Polynomial fitting, Cressmann's scheme & Optimum interpolation scheme.

Objectively analysed values of different field variables at the grid points are most likely to contain error. Errors in the data may physically be interpreted as an imbalance between different forces, caused due to interpolation method used in objective analysis. This imbalance may results in the generation of spurious waves, which may amplify with time and propagate into the forecast region and thus may spoil the forecast. Thus it is essential to eliminate such spurious wave by removing such errors from the objectively analysed data. This is known as initialisation. Broadly there are two types of initialisation scheme, viz., static initialisation and dynamic initialisation.